VORTICITY IN GRADIENT FLOW

BY

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PREFACE

The purpose of this research was to find solutions of the differential equation of vorticity in gradient flow and to investigate the application of such solutions to the forecasting of upper air trajectories.

This work was conducted at the U. S. Naval Postgraduate School, Monterey, California, during the period December 1950 to June 1951.

I wish to express my appreciation to Professor W. D. Duthie for advice and guidance throughout.

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TABLE OF SYMBOLS AND ABBREVIATIONS

$_o$	Subscript refers to value at initial point
$\underline{V}$	Gradient Wind Vector
$\underline{V}_{gs}$	Geostrophic Wind Vector
ζ_a	Absolute Vorticity
ζ	Relative Vorticity
∇	Horizontal Differential Operator
∇_p	Differential Operator for Constant Pressure Surface
ρ	Density
g	Acceleration due to Gravity
λ	Coriolis Parameter
K_t	Curvature of Trajectory
K_s	Curvature of Streamlines
C	Speed of Moving Pressure System
ψ	Angle Between Path of System and Wind or Streamline
T	Absolute Temperature
$\frac{\partial V}{\partial n}$	Horizontal Shear of Gradient Wind
$\epsilon = \frac{1}{\zeta_a}\left\{\frac{\partial V}{\partial n} - (K_t - K_s)V\right\}$	
$\nu = \frac{1}{\zeta_a}\frac{\partial V}{\partial n}$	
$\frac{\partial}{\partial t}$	Local Variation of Property with respect to Time

I. INTRODUCTION

The possibility of making accurate mechanical forecasts of weather phenomena by straightforward solutions of the basic equations governing atmospheric motions has long intrigued meteorologists. Theoretically, such a solution to the problem is possible but there are so many varied and complex forces at work in the atmosphere that all attempts so far to effect a complete solution to these equations have met with failure.

One of the more important difficulties in the way of reaching satisfactory solutions has been the non-linearity of the equations and no standard methods are available to integrate nonlinear partial differential equations. Therefore the introduction of the so-called "perturbation" method was made which consists essentially in linearizing the equations by assuming meteorological quantities having basis values unchanging with time, plus small perturbation values whose second order terms can be neglected.

One approach to quantitative forecasting has been the linearization and solution of the absolute vorticity equation. This was introduced by Rossby [10], who obtained a solution for the motion of sinusoidal waves of infinite lateral extent in a horizontal plane and found that the velocity of propagation of such waves is given by the well known formula

$$C = U - \frac{\beta L^2}{4\pi^2}$$

where C is the velocity of the waves toward the East, U is average speed of the westerly current, L is wave length and β is the meridional rate of change of the Coriolis parameter. A considerable literature has grown out of the

interpretation of this formula on the classification of large scale atmospheric systems in terms of the so-called zonal index, which is a measure of the strength of the zonal current.

Haurwitz extended Rossby's method and obtained solutions for the motion of waves of finite lateral extent on a horizontal plane [5] and on a sphere [6] . Craig [1] obtained solutions of the vorticity equations in complete form without linearization for the plane and the sphere. These solutions differed from those of Haurwitz by the absence of a zonal velocity term. Neamtan [7] then made a treatment of the vorticity in the same manner as Craig, viz by the use of stream functions and obtained identical solutions to those of Haurwitz. He showed that Craig's solutions were incomplete, causing absence of the zonal index term.

Forsythe [3] developed formulas for the speed of propagation of waves with changing shape by use of the scalar relative vorticity as an identifiable property. However, he did not make any test on his formulas to determine their usefulness in practical forecasting.

Rossby and co-workers [11] supplemented his initial formula based on the conservation of absolute vorticity with a more general technique which would be applicable to arbitrary initial streamline pattern so that the difficulty of defining a prevailing wave length would be circumvented.

All of the above solutions of the vorticity equations were predicated on the assumption of frictionless, non-divergent, and autobarotropic flow. Further, in order to obtain a solution for relative vorticity which could be useful in forecasting trajectories it was necessary for Rossby [11] to assume no horizontal shear of the wind and stationary pressure systems.

In this treatment, a general solution of the non-linear differential equation of the vertical component of absolute vorticity will be obtained without assuming non-divergent flow or barotropic conditions. A solution is obtained for (1) a surface of constant height, and (2) a constant pressure surface. A discussion of mean-value constants for the wind shear term and for the effect of moving pressure systems, by means of which the vorticity equation could be integrated without disregarding these two terms, is included. A chapter is devoted to examining this solution with regard to its adaptation to forecasting of upper-air flow pattern by the trajectory method. In addition, special forms of the vorticity equation resulting from various assumptions are developed and discussed.

Gradient flow is assumed throughout this investigation. The use of gradient flow permits a special handling of the divergence term. The wind of course is not always gradient even in the free atmosphere but is closely approximated by the gradient wind at elevations greater than 1000 meters above the ground. However, under conditions of rapidly changing pressure gradient this close relationship between observed wind and gradient wind is greatly modified, and is due to the fact that the motion is not under balanced forces. Under these conditions there is a velocity component along the isallobaric gradient. Consequently the use of gradient flow in the vorticity equation would generally be in error.

II. SOLUTION OF THE VORTICITY EQUATION FOR GRADIENT FLOW

In obtaining solutions to the differential equation of the vertical component of absolute vorticity as few assumptions as possible will be employed. Initially, only the following assumptions are made:

a. Friction is negligible

b. Vertical velocities are negligible.

Other restrictions will be added later to obtain solutions of particular cases. The general equation of vorticity to be solved, which is due to Bjerknes, is:

$$\frac{d\zeta_a}{dt} = -\zeta_a \nabla\cdot\underline{V} + \lambda\frac{\nabla T}{T}\cdot\underline{V}_{gs} \quad . \tag{2.1}$$

It is noted that this equation consists of two terms, a term which represents the horizontal divergence of the gradient wind and a solenoidal term which expresses the component of the geostrophic wind along the gradient of temperature.

Since $\underline{V}_{gs} = -\frac{1}{\rho\lambda}\nabla p \times \mathbb{k}$,

$$\frac{d\zeta_a}{dt} = -\zeta_a \nabla\cdot\underline{V} - \frac{\nabla T}{T}\cdot\alpha(\nabla p\times \mathbb{k}) \quad . \tag{2.2}$$

According to Taylor [12] the following expression for $\nabla\cdot\rho\underline{V}$ can be derived:

$$\nabla\cdot\rho\underline{V} = -\nabla p\times\mathbb{k}\cdot\nabla(\lambda+K_t V)^{-1} + (\lambda+K_t V)^{-1}\nabla\cdot\nabla p\times\mathbb{k}$$

which, solved for $\nabla\cdot\underline{V}$, yields

$$\nabla\cdot\underline{V} = -\frac{\nabla p\times\mathbb{k}\cdot\nabla(\lambda+K_t V)^{-1}}{\rho} - \frac{\underline{V}\cdot\nabla\rho}{\rho} \quad .$$

Now if we put this value of $\nabla\cdot\underline{V}$ in equation (2.2) and divide by $\mathcal{S}_a$ we obtain:

Error in sign discovered by W T Duthie 4/12/55

$$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = +\alpha\,\underline{V}\cdot\nabla\rho + \alpha\,\nabla P\times\mathbb{k}\cdot\nabla(\lambda+K_tV)^{-1} - \frac{\nabla T}{T}\cdot\frac{\alpha(\nabla P\times\mathbb{k})}{\mathcal{S}_a} \quad . \quad (2.3)$$

Now $\dfrac{\nabla\rho}{\rho} = \dfrac{\nabla P}{P} - \dfrac{\nabla T}{T}$,

so that

$$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = +\underline{V}\cdot\left(\frac{\nabla P}{P} - \frac{\nabla T}{T}\right) + \alpha\,\nabla P\times\mathbb{k}\cdot\nabla(\lambda+K_tV)^{-1} - \frac{\nabla T}{T}\cdot\frac{\alpha(\nabla P\times\mathbb{k})}{\mathcal{S}_a} \quad . \quad (2.4)$$

$\underline{V}\cdot\nabla P = 0$ because the vectors are perpendicular. Hence

$$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = -\underline{V}\cdot\frac{\nabla T}{T} + \alpha\,\nabla P\times\mathbb{k}\cdot\nabla(\lambda+K_tV)^{-1} - \frac{\nabla T}{T}\cdot\frac{\alpha(\nabla P\times\mathbb{k})}{\mathcal{S}_a} \; .$$

Now if we make the substitution

$$(\lambda+K_tV)\underline{V} = -\alpha\,\nabla P\times\mathbb{k}$$

we obtain

$$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = -\underline{V}\cdot\frac{\nabla T}{T} \mp (\lambda+K_tV)\underline{V}\cdot\nabla(\lambda+K_tV)^{-1} + \frac{\nabla T}{T}\cdot\frac{(\lambda+K_tV)\underline{V}}{\mathcal{S}_a} \; .$$

Rearranging and collecting terms

$$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = -\underline{V}\cdot\left\{\frac{\nabla T}{T}\left(1\mp\frac{(\lambda+K_tV)}{\mathcal{S}_a}\right) \mp (\lambda+K_tV)\nabla(\lambda+K_tV)^{-1}\right\}$$

or, $$\frac{1}{\mathcal{S}_a}\frac{d\mathcal{S}_a}{dt} = -\underline{V}\cdot\left\{\left[1\mp\frac{(\lambda+K_tV)}{\mathcal{S}_a}\right]\nabla\ln T + \nabla\ln(\lambda+K_tV)\right\} \; .$$

Integrating along the path of the particle,

$$\int d(\ln\mathcal{S}_a) = -\int\left(1\mp\frac{(\lambda+K_tV)}{\mathcal{S}_a}\right)\nabla\ln T\cdot\underline{V}\,dt \mp \int\nabla\ln(\lambda+K_tV)\cdot\underline{V}\,dt$$

and since $\underline{V}\,dt = d\underline{r}$,

$$\int d(\ln \zeta_a) = -\int \left(1 \mp \frac{\lambda + K_t V}{\zeta_a}\right) d(\ln T) + \int d[\ln(\lambda + K_t V)] . \tag{2.5}$$

Now $\zeta_a = \lambda + K_s V + \frac{\partial V}{\partial n}$, where $\frac{\partial V}{\partial n}$ is the horizontal shear of the gradient wind, and the relation of curvature of streamlines to curvature of trajectories is $K_t V = K_s V + \frac{\partial \psi}{\partial t}$. Hence equation (2.5) can be written, after integrating and rearranging terms,

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{T^2 \left[\zeta_a + (K_t - K_s)V - \frac{\partial V}{\partial n}\right]_0}{T_0^2 \left[\zeta_a + (K_t - K_s)V - \frac{\partial V}{\partial n}\right]} \exp - \int \frac{\frac{\partial V}{\partial n} - (K_t - K_s)V}{\zeta_a}\, d(\ln T), \tag{2.6}$$

where the subscript $_0$ indicates the initial values of the variable quantities.

We note from equation (2.6) that on following a particle along a path from an initial point to some other point on the path, the ratio of their absolute vorticities is equal to the ratio of their temperatures times the inverse ratio of terms involving curvature and shear times an exponential term.

Equation (2.6) was derived without assuming either zero divergence or a barotropic atmosphere. Furthermore, no restrictions have yet been placed on the shear of the wind or changing streamlines.

Now if we let $\epsilon = \frac{1}{\zeta_a}\left\{\frac{\partial V}{\partial n} - (K_t - K_s)V\right\}$ let $\bar{\epsilon} = \dfrac{\frac{\partial V}{\partial n} - (K_T - K_S)V}{\zeta_a}$,

and $\bar{\epsilon} \doteq$ mean value of ϵ along the path, then

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^{2-\bar{\epsilon}} \frac{(\zeta_a - \zeta_a \bar{\epsilon})_0}{(\zeta_a - \zeta_a \bar{\epsilon})} = \left(\frac{T}{T_0}\right)^{2-\bar{\epsilon}} \frac{\zeta_a(1-\bar{\epsilon})_0}{\zeta_a(1-\bar{\epsilon})} . \tag{2.7}$$

$$\frac{\zeta_a}{\zeta_{a0}} = \frac{\lambda + K_T V}{(\lambda + K_T V)_0} \left(\frac{T}{T_0}\right)^{\bar{\epsilon}}$$

If we assume a stationary pressure system, $\frac{\partial \psi}{\partial t} = 0$, as do most investigations of vorticity, then equation (2.6) becomes

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^2 \frac{\left(\zeta_a - \frac{\partial V}{\partial n}\right)_0}{\left(\zeta_a - \frac{\partial V}{\partial n}\right)} \exp - \int \frac{1}{\zeta_a} \frac{\partial V}{\partial n} \, d(\ln T) \qquad (2.8)$$

1. Discussion of Orders of Magnitude of the Mean Value Constant in Equation (2.8) for Stationary Systems.

If we compare the relative order of magnitude of the terms contained in the exponential part of equation (2.8) some conclusions can be reached regarding the value of the term $\frac{1}{\zeta_a} \frac{\partial V}{\partial n}$ so that a constant can be assigned to this term which will represent its average value over most mid-latitude paths. With such a constant the exponential terms can then be integrated.

An examination of 700 mb. charts appears to demonstrate a radius of curvature of most waves of the order of 600 miles. If we take an average value of 20 knots for the wind velocity the order of magnitude for the curvature term will be 10^{-5} sec^{-1}. Estimating an order of magnitude for the shear term in a similar manner we use for an average shear a change of 20 knots in a distance of 600 miles. This gives an order of magnitude of 10^{-5} sec^{-1} for the shear term. An average value for the coriolis force term λ would be the value at 45° latitude. At this latitude the coriolis term is almost exactly equal to 10^{-4} sec^{-1}. Therefore, the order of magnitude for the curvature and shear terms will closely approximate 10^{-5} sec^{-1} and for the coriolis term will be 10^{-4} sec^{-1}. It is seen from this qualitative comparison that the coriolis force is about ten times greater than the other two terms in influencing changes in the absolute vorticity.

When these values are placed in the expression $\frac{1}{\zeta_a}\frac{\partial V}{\partial n}$ we obtain the value of $-1/9$ when the shear term is negative and the value of $1/11$ when the shear is positive.

Therefore the general solution (2.8) can now be written as

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^{2+\nu} \frac{\left(\zeta_a - \frac{\partial V}{\partial n}\right)_0}{\left(\zeta_a - \frac{\partial V}{\partial n}\right)} \tag{2.9}$$

where $\nu = 1/9$ or $-1/11$.

The $\left(\frac{T}{T_0}\right)^{2+\nu}$ term will then be $\left(\frac{T}{T_0}\right)^{2\,1/9}$ or $\left(\frac{T}{T_0}\right)^{1\frac{10}{11}}$. $(2+\nu)$ differs from 2.0 by about 5%. Now, since the ratio of $\left(\frac{T}{T_0}\right)$ will usually be of the order of unity the error involved in using simply $\left(\frac{T}{T_0}\right)^2$ instead of $\left(\frac{T}{T_0}\right)^{2+\nu}$ will only be about 1%.

Hence, by neglecting the exponential term represented by ν, equation (2.6) can be written simply as

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^2 \frac{\left(\zeta_a - \frac{\partial V}{\partial n}\right)_0}{\left(\zeta_a - \frac{\partial V}{\partial n}\right)} \tag{2.10}$$

2. Discussion of Constants in the Equation for Moving Pressure Systems.

All treatments of the vorticity equation appearing in the literature so far have ignored the effects of difference between the curvatures of streamlines and trajectories $\frac{\partial \psi}{\partial t}$, and have assumed a steady state condition. This implies stationary pressure systems which of course is not the prevailing case in nature. The difficulty of handling $\frac{\partial \psi}{\partial t}$ mathematically has been responsible for neglecting this term.

Perhaps some conclusions can be drawn, however, from an examination of the range of variation of $V(K_t - K_s)$ so that a mean value constant can be assigned to ϵ with the result that equation (2.7) can be integrated.

From the relationship of trajectory curvature and streamline curvature according to Petterssen [8, p. 225], $K_t = K_s(1 - \frac{c}{V}\cos\psi)$, we can solve for $V(K_t - K_s)$ so that

$$V(K_t - K_s) = K_s C \cos\psi \ .$$

Consider now a wave shaped system which is more or less symmetrical about a latitude circle and moving eastward with speed C. Along the trough which is to the right of the path of the system Cos $\psi > 0$ and is unity at bottom of trough. Along the ridge where K_s is anticyclonic to left of path Cos $\psi < 0$. If we choose average values of ± 0.7 for Cos ψ, the sign depending on whether along a trough or ridge, and take average values of 1000 km and 20 kts for R_s and C respectively then $V(K_t - K_s) \sim 10^{-5}$ order of magnitude.

From previous considerations of orders of magnitude for the shear and absolute vorticity we arrived at values of 10^{-5} sec^{-1} and 10^{-4} sec^{-1} respectively for two terms. Therefore the final order of magnitude for will be two possible values:

$$\bar{\epsilon} \sim \frac{1}{10^{-4}}\left(\pm 10^{-5} - 10^{-5}\right) .$$

If the shear is negative, then $\bar{\epsilon} \sim -0.2$, and if shear is positive, $\bar{\epsilon} \sim 0$.

Finally, substituting these two possible values in equation (2.7) we obtain

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^{2.2} \frac{(1.2\zeta_a)_0}{(1.2\zeta_a)}, \quad \bar{\epsilon} \sim -0.2$$

and

$$\frac{\zeta_a}{\zeta_{a_0}} = \left(\frac{T}{T_0}\right)^{2} \frac{\zeta_{a_0}}{\zeta_a}, \quad \bar{\epsilon} \sim 0.$$

Therefore, under the conditions stated above the existence of positive or cyclonic wind shear cancels out the effect of the $\frac{\partial \psi}{\partial t}$ term and with anticyclonic or negative shear the correction term ϵ is significant. The opposite effects would occur in the case of retrograde waves or in those parts of closed pressure systems where the wind blows in a direction opposite to the direction of motion of the system.

III. SPECIAL CASES OF THE GENERAL SOLUTION OF THE VORTICITY EQUATION FOR STATIONARY SYSTEMS

Certain simplifying assumptions can now be made with regard to the various terms of the general solution (2.8). The special cases which will be discussed here are:

1. Geostrophic Flow
2. Wind Flow Parallel to Isotherms
3. No Horizontal Wind Shear
4. Combinations of the Above Three Cases.

1. Case of Geostrophic Flow.

Geostrophic flow prevails where there exist straight isobars or contours, hence the curvature term becomes zero and equation (2.8) becomes:

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{T^2}{T_0^2}\frac{\lambda_0}{\lambda}\, e^{-\int \frac{1}{\zeta_a}\frac{\partial V}{\partial n}\, d(\ln T)} \quad . \tag{3.1}$$

However, as demonstrated in Chapter II, the exponential term can be neglected with only a small error involved so that for the geostrophic flow case equation (3.1) can be written as

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{T^2 \lambda_0}{T_0^2 \lambda} \quad . \tag{3.2}$$

Equation (3.2) can be written as a quadratic and solved by the quadratic formula so that

$$\zeta_a = \frac{1}{2}\left\{\frac{\partial V}{\partial n} \pm \sqrt{\left(\frac{\partial V}{\partial n}\right)^2 + \left(\frac{T}{T_0}\right)^2 c}\right\} \tag{3.3}$$

where $c = \left(\lambda + \frac{\partial V}{\partial n}\right)_0 \lambda_0$.

Equation (3.3) can now be expanded binomially:

$$\zeta_a = \tfrac{1}{2}\frac{\partial v}{\partial n} \pm \tfrac{1}{2}\left\{\frac{\partial v}{\partial n} + \tfrac{1}{2}\left(\frac{\partial v}{\partial n}\right)^{3/2}\left(\frac{T}{T_0}\right)^2 c - \tfrac{1}{8}\left(\frac{\partial v}{\partial n}\right)^{1/2}\left(\frac{T}{T_0}\right)^4 c^2 + \right.$$
$$\left. + \tfrac{1}{16}\left(\frac{\partial v}{\partial n}\right)^{-1/2} c^3 \left(\frac{T}{T_0}\right)^6 + \cdots\right\}.$$

All terms beyond the first in this binomial expansion are of the order of 10^{-8} or smaller and which, because of a much smaller order of magnitude than for the shear term, can be neglected so that for the case of geostrophic flow the absolute vorticity can be expressed simply by $\zeta_a = \tfrac{1}{2}\frac{\partial v}{\partial n} + \tfrac{1}{2}\frac{\partial v}{\partial n} = \frac{\partial v}{\partial n}$.

Inasmuch as the absolute vorticity cannot be zero except for the very improbable case of geostrophic flow with no shear and at the equator, the choice of sign on the second term in (3.3) must be positive.

For practical application on the weather chart in determing how such flow would be affected by various types of movement, the relative vorticity is examined. For geostrophic flow, (3.2) can be written

$$\zeta = \left(\frac{T}{T_0}\right)^2 \frac{\lambda_0}{\lambda}\left(\zeta + \lambda\right)_0 - \lambda \quad .$$

A particle moving northward would therefore undergo a decrease in relative vorticity or be turned anticyclonically even though there may be a slight positive contribution from the temperature term. Southward flow would produce the opposite effect.

A particle moving along a parallel of latitude would be influenced only by the change in temperature. If moving toward higher temperature, there will be an increase in relative vorticity and particle will curve to the left.

2. Case of Wind Flow Parallel to Isotherms.

The wind is blowing parallel to the isotherms in this case, which implies a barotropic atmosphere. Such conditions prevail when systems are thermally symmetrical. In this case equation (2.8) becomes

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{(K_s V + \lambda)_0}{(K_s V + \lambda)} \tag{3.4}$$

and the exponential term becomes unity. Now, on following a particle from initial point or path to some other point the absolute vorticity changes as the inverse ratio of their respective $(K_s V + \lambda)$ terms.

If we now expand equation (3.4) a quadratic is obtained which can be written as $\zeta_a^2 - \zeta_a \frac{\partial V}{\partial n} - C$, where $C = \left(K_s V + \lambda + \frac{\partial V}{\partial n}\right)_0 (K_s V + \lambda)_0$, the initial point values. Solving this equation by means of the quadratic formula gives

$$\zeta_a = \tfrac{1}{2}\left(\frac{\partial V}{\partial n} \pm \sqrt{\left(\frac{\partial V}{\partial n}\right)^2 + c}\right) . \tag{3.5}$$

Equation (3.5) can now be expanded in a binomial expansion giving in a manner similar to Case 1:

$$\zeta_a = \frac{\partial V}{\partial n} . \tag{3.6}$$

The choice of the positive sign for the second term is discussed under Case 1.

Equation (3.4) can be now written as relative vorticity

$$\zeta = \frac{(K_s V + \lambda)_0 \left(K_s V + \lambda + \frac{\partial V}{\partial n}\right)_0}{(K_s V + \lambda)} - \lambda .$$

3. Case of No Horizontal Wind Shear.

If the horizontal shear of the wind is either assumed to be zero or is neglected then the general equation (2.8) simplifies to

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{T^2 (K_s V + \lambda)_0}{T_0^2 (K_s V + \lambda)} \quad . \tag{3.7}$$

This can be written as

$$\frac{\zeta_a^2}{\zeta_{a_0}^2} = \frac{T^2}{T_0^2} \quad ; \quad \frac{\zeta_a}{\zeta_{a_0}} = \frac{T}{T_0} \quad . \tag{3.8}$$

We see then that for the case of no shear the ratio of the absolute vorticity at two points is equal to the ratio of their respective absolute temperatures. It is also to be noted that this relationship which involves no assumptions as to divergence of the wind or barotropy differs from Rossby's formula as applied to the forecasting of particle trajectories only by the presence of the temperature term.

If we express the vorticity as $K_s V + \lambda$ and choose an inflection latitude, as did Rossby and co-workers [11] , at which the curvature is zero, then equation (3.8) can be expressed as

$$K_s V = \frac{T}{T_0} \lambda_0 - \lambda \tag{3.9}$$

which corresponds to Rossby's formula

$$K_s V = \lambda_0 - \lambda \quad .$$

Equation (3.9) could be applied in a manner entirely analogous to the technique used by Rossby in forecasting air particle trajectories. The coriolis force at any point from the inflection latitude can be expressed

as a function of distance from such latitude so that $\lambda = \beta y + \lambda_0$ where β is the rate of change of coriolis force with latitude and is considered constant over the latitude range concerned. Then (3.9) can be written as

$$K_s V = \frac{T}{T_0}\lambda_0 - (\beta y + \lambda_0) \,. \tag{3.10}$$

If we express the curvature in terms of a second order differential equation of its y coordinate or distance from the inflection latitude we obtain after substituting this in (3.10)

$$\frac{V y''}{(1+y'^2)^{3/2}} = \lambda_0\left(\frac{T}{T_0} - 1\right) - \beta y \,.$$

This differential equation could perhaps be solved by elliptic integrals if a proper constant is chosen for the temperature term so that a value of y coordinate would be obtained as a function of the intersection angle of particle path with inflection latitude.

However, a qualitative interpretation of equation (3.10) can be made showing effects on the relative vorticity with varying paths without actually solving the differential equation. If the particle is moving northward initially, the effect of coriolis force is to decrease the relative vorticity and increasing temperature tends also to increase the vorticity. However, now assuming decrease in temperature northward the temperature term in general is of much smaller magnitude than βy , hence the particle curves anticyclonically but to a slightly greater extent than the path resulting from coriolis considerations only. If the temperature increases northward along the path the effect is to decrease the effect of increasing coriolis force so that relative vorticity is decreased to a lesser extent than when

the coriolis effect is considered alone. Therefore the path would curve less anticyclonically than otherwise.

For a southward moving current the converse of all the above statements would hold.

4. Combinations of the Three Preceding Cases.

a. Geostrophic Flow with no Wind Shear.

Under such conditions of wind flow equation (2.8) becomes simply

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{T^2}{T_0^2}\,\frac{\lambda_0}{\lambda}$$

which can be simplified to

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{\lambda}{\lambda_0} = \frac{T}{T_0}$$

or the change of vorticity, which is now due simply to change of coriolis force, is directly proportional to the change of respective absolute temperatures. It is also interesting to note that the flow in this case implies a gradient of temperature from South to North, which of course, is not true in nature except under local conditions.

b. Geostrophic Flow with a Barotropic Atmosphere.

In this case the general equation (2.8) can be written

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{\lambda_0}{\lambda}$$

or

$$\frac{\left(\lambda + \frac{\partial V}{\partial n}\right)}{\left(\lambda + \frac{\partial V}{\partial n}\right)_0} = \frac{\lambda_0}{\lambda} .$$

c. A Barotropic Atmosphere with no Wind Shear.

Under these conditions equation (2.8) becomes

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{(K_s V + \lambda)_0}{(K_s V + \lambda)} .$$

This can be written as

$$(K_s V + \lambda) = (K_s V + \lambda)_0 .$$

This result is identical in form to that used by Rossby and co-workers [11] in forecasting constant vorticity trajectories with the significant difference that no restrictions have been placed on divergence or convergence in this development. Therefore, the same technique as used by Rossby in forecasting particle trajectories could be applied to the gradient wind using identical relationships as:

$$K_s V = -\beta y \ ,$$

where β = rate of change of coriolis force and (y) equals distance from an inflection latitude at which relative vorticity and curvature are zero. Solutions of the second order differential equation for the maximum y co-ordinate as a function of the intersection angle of the path with the inflection latitude would be identical to those of Rossby [11] or Fultz [4].

The verification of forecasts of such computed trajectories should be more accurate than that of Rossby because of the added refinement of no restriction on divergence and convergence.

d. Combination of all Three Special Cases -- Geostrophic Flow, No Wind Shear, and Barotropy.

These conditions would give a form for the vorticity equation thus

$$\frac{\zeta_a}{\zeta_{a_0}} = \frac{\lambda_0}{\lambda} ,$$

which gives the rather trivial result

$$\lambda = \lambda_0 \ .$$

However, absolute vorticity is again shown to be constant although only a function of latitude. Furthermore, zonal flow is implied by this equality of coriolis terms.

IV. SOLUTION OF THE VORTICITY EQUATION FOR A CONSTANT PRESSURE SURFACE

The equation of absolute vorticity along a surface of constant pressure is somewhat more simplified than along a constant height surface because the solenoid term is absent. This is not surprising since an isobaric surface obviously cannot be intersected by solenoids.

Thus, equation (2.2) when applied to an isobaric surface involves only a divergence term:

$$\frac{d \zeta_a}{dt} = - \zeta_a \nabla \cdot \underline{V} \quad . \tag{4.1}$$

We can now express this divergence in terms of coriolis force, curvature and velocity in a manner entirely analogous to that for a constant height surface.

Starting with the expression for the gradient wind:

$$\underline{V} = \underline{V}_{gs} - \frac{(K_t V)\underline{V}}{\lambda}$$

which we can express for an isobaric surface as

$$\underline{V} = -g/\lambda \, \nabla_p z \times \mathbb{k} - \frac{(K_t V)\underline{V}}{\lambda} \quad . \tag{4.2}$$

Now multiplying by λ and rearranging:

$$-g \nabla_p z \times \mathbb{k} = \lambda \underline{V} + K_t V \underline{V}$$

solving for $\underline{V}$ gives

$$\underline{V} = -g \nabla_p z \times \mathbb{k} (\lambda + K_t V)^{-1} .$$

Taking the divergence of both sides gives

$$\nabla_p \cdot \underline{V} = -g \nabla_p z \times \mathbb{k} \cdot \nabla_p (\lambda + K_t V)^{-1} - (\lambda + K_t V)^{-1} \nabla_p \cdot g \nabla_p z \times \mathbb{k}$$

which reduces to

$$\nabla_p \cdot \underline{V} = -g \nabla_p Z \times \mathbb{k} \cdot \nabla_p (\lambda + K_t V)^{-1} \tag{4.3}$$

since $\nabla_p \cdot \nabla_p \times \mathbb{k} = 0$.

It is seen that equation (4.3) gives the divergence of the gradient wind as the scalar product of a vector parallel to the isohypses and the gradient of the quantity $(\lambda + K_t V)^{-1}$.

Upon substituting this expression for the divergence in (4.1) and rearranging we get

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$$\frac{1}{\mathcal{S}_a} \frac{d \mathcal{S}_a}{dt} = + g \nabla_p Z \times \mathbb{k} \cdot \nabla_p (\lambda + K_t V)^{-1}.$$

Since $-g \nabla_p Z \times \mathbb{k} = + \underline{V} (\lambda + K_t V)$ we can write, after performing indicated differential operations:

$$\frac{1}{\mathcal{S}_a} \frac{d \mathcal{S}_a}{dt} = + \underline{V} \cdot \frac{\nabla_p (\lambda + K_t V)}{(\lambda + K_t V)}$$

which can be expressed as the integral

$$\int d(\ln \mathcal{S}_a) = + \int \underline{V} \cdot \nabla \ln (\lambda + K_t V)\, dt. \tag{4.4}$$

Now since $\frac{d\underline{r}}{dt} = \underline{V}$ and $\nabla_a \cdot d\underline{r} = da$ we can write (4.4) as

$$\int d(\ln \mathcal{S}_a) = + \int d \ln (\lambda + K_t V)$$

which is readily integrable to

$$\ln \frac{\mathcal{S}_a}{\mathcal{S}_{a_0}} = + \ln \frac{(\lambda + K_t V)}{(\lambda + K_t V)_0},$$

and

$$\frac{\mathcal{S}_a}{\mathcal{S}_{a_0}} = \frac{(\lambda + K_t V)_0}{\lambda + K_t V}. \tag{4.5}$$

Equation (4.5) is identical with (3.4) which was obtained for a constant level surface by assuming a barotropic atmosphere. However, in developing equation (4.5) no such assumption was necessary. For equation (4.5) to be applicable, however, the flow must be assumed to take place along an isobaric surface. Therefore its application would be highly limited because of the improbability of isobaric flow.

Now equation (4.5) can be rearranged as a quadratic and expanded binomially in a manner similar to equation (3.4) and which gives identical results:

$$(\zeta_0)_p = \frac{\partial V}{\partial n} .$$

As a suggested extension to this work solutions of the vorticity equation could be carried out in an entirely analogous manner for isentropic surfaces and would have wider application than the solution for a constant pressure surface because of the greater prevalence of flow along isentropic surfaces.

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